Quantum fluids in the Kaehler parametrization

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Abstract

In this paper we address the problem of the quantization of the perfect relativistic fluids formulated in terms of the Kähler parametrization. This fluid model describes a large set of interesting systems such as the power law energy density fluids, Chaplygin gas, etc. In order to maintain the generality of the model, we apply the BRST method in the reduced phase space in which the fluid degrees of freedom are just the fluid potentials and the fluid current is classically resolved in terms of them. We determine the physical states in this setting, the time evolution and the path integral formulation.

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1 Introduction

The recent interest in formulating the perfect relativistic fluids in the framework of the Lagrangian and Hamiltonian canonical formalisms has been motivated by the necessity of studying the fluid regime of systems with more complex symmetries such as the non-abelian fluids [1], the fluid model of D-brane [2] and NS-brane systems [3], supersymmetric fluids [4] and non-commutative fluids [5]. Each of these mathematical models is interesting per se being the source of new problems of the mathematical physics and because of its relation with specific phenomenology. The physical phenomena that can be modeled by the fluid model systems are characterized by degrees of freedom that have a larger scale than the typical scale of the corresponding system. Usually, these two scales are in the relationship of the macroscopic versus microscopic type. This defines the fluid limit in which the action functionals are constructed. However, once the action is found, the reverse way can be taken to inquire about the quantum properties of the fluids. When this is possible, methods and tools of the quantum field theory can be used to explore the quantum fluids.

The study of the classical and quantum dynamics of the perfect relativistic fluids in the Lagrangian and Hamiltonian formalism meets the obstruction created by the Casimir-like invariants [2] that prevent finding the inverse of the symplectic form defined in the phase space of the degrees of freedom of the fluid. These obstructions can be removed by parameterizing the fluid velocity such that the invariant be given by a surface integral. Then the obstruction no longer contribute to the bulk physics and the construction of the Lagrangian and Hamiltonian functionals is possible. There are at least two parametrizations that satisfy the above property: the Clebsch parametrization formulated in terms of three real fluid potentials $\theta(x)$, $\alpha(x)$ and $\beta(x)$, respectively, [2, 6, 7] and the Kähler parametrization in which the potentials are $\theta(x)$ which is real and $z(x)$ and $\bar{z}(x)$ which are complex conjugate to each other, respectively [8]. While the classical dynamics can be successfully determined in each of the two descriptions, the Kähler parametrization is particularly interesting because it displays an infinite number of symmetries related to the reparametrization of the complex manifold of the two complex potentials. This model was generalized to supersymmetric fluids [8], superhydrodynamics [9], conformal fluids [10], metafluids [11], supersymmetric fluids in AdS_5 [12] and noncommutative fluids [13].

The perfect relativistic fluids in the Kähler parametrization form a large class of systems characterized by two arbitrary smooth functions: $K(z,\bar{z})$ which is the Kähler potential of the metric on two dimensional complex surface of coordinates $z(x)$ and $\bar{z}(x)$ and $f(\rho)$ which is a function of the fluid density of mass ρ that characterizes the equation of state. Different choices of $K(z,\bar{z})$ and $f(\rho)$ correspond to different perfect fluids. The models described by the fluids from this class are the ideal fluid, the power law energy density fluids, the Chaplygin fluid, etc. Therefore, it is certainly interesting to understand the quantum systems that correspond to these models. In [14] we have quantized a simple fluid characterized by the Kähler potential of the complex plane $K(z,\bar{z}) = z\bar{z}$ and by the function $f(\rho) \sim \rho^2$ by applying the canonical quantization methods of the Quantum Field Theory.

In this paper, we address the more general question whether it is possible to quantize the general class of the relativistic perfect fluids in the Kähler parametrization, that is, for arbitrary $K(z,\bar{z})$ and $f(\rho)$. The classical analysis of the fluid variables was done in [8] where it was shown that the spatial components of the fluid current j_μ can be expressed in terms of the fluid potentials $\{\theta(x), z(x), \bar{z}(x)\}$ while the time-like component of the current is equal to the momentum $\pi_\theta(x)$. It was concluded in [8] that the classical dynamical degrees of freedom span the *reduced phase space* $\Gamma = \{\theta, z, \bar{z}, \pi_\theta, \pi_z, \pi_{\bar{z}}\}$. The physical phase space Γ^* is the reduced

phase space acted upon by two second-class constraints $\Omega_\alpha, \alpha = 1, 2$. This setting is very similar to the one encountered in the study of the gauge theories with second class constraints for which a variety of quantization methods have been developed. Therefore, it is natural to attempt to quantize the relativistic fluid in the reduced phase space. To this end, we are going to use the BRST method in the Hamiltonian formulation [15]. Since the system has second class constraints, our quantization is similar to the BFV formulation [16, 17, 18, 19]. By exploiting this similarity, we determine the effective Hamiltonian for the perfect relativistic fluid, establish a set of operatorial equations from which the quantum states can in principle be computed and discuss the time evolution of the system. In our quantization scheme we extended the reduced space of the fluid fields to include BRST ghost variables associated to the constraints. Consequently, the states belong to a vector space with an undefined metric. Thus, the states must belong to the extended inner product space V that contains in a subspace $V_{phys} \subset V$ the physical states. Like in the case of the gauge fields, the quantization of the degrees of freedom can be performed with the positive or non-definite states, like in the inner product space formalism [20, 21, 22, 23, 24].

The paper is organized as follows. In Section 2 we briefly review the classical theory of the relativistic fluid in the Kähler parametrization. In particular, we give the reduced phase space and the two second class constraints that act on it. This classical structure is suitable for the BRST quantization in the inner space formalism which is done in Section 3. In particular, we determine here the quantum physical states as singlets of the BRST operator constructed from combinations (involutions) of the BRST doublets and triplets that characterize the states of the fluid. The physical states should be invariant under the transformations of the set of operators which has the structure of $U(1)^4$ that contain two reparametrization $U(1)$ subgroups and two rotation ones. We calculate the unitary invariant states under the reparametrization and construct the more general BRST singlets out of them. In Section 4 we determine the BRST invariant Hamiltonian that generates the time evolution of the quantum states. Using the results from Section 3, we obtain the path integral of the quantum relativistic fluid and discuss the transition probabilities. The last section is reserved to discussions.

2 Relativistic fluid in the Kähler parametrization

The dynamics of the perfect relativistic fluids in the Minkowski space-time of metric $\eta_{\mu\nu} = (-, +, +, +)$ can be obtained by applying the principle of the least action to the following Lagrangian density [8]

$$\mathcal{L}[j^\mu, \theta, \bar{z}, z] = -j^\mu \left(\partial_\mu \theta + i\partial K \partial_\mu z - i\overline{\partial} K \partial_\mu \bar{z}\right) - f(\rho). \tag{1}$$

Here, $j^\mu = \rho u^\mu$ is the current four-vector, ρ is the density of mass, $u^\mu = dx^\mu/d\tau$ is the velocity four-vector of the fluid element with $u^2 = -1$ and τ is the proper time along the flow line of the current. The three fluid potentials in the Kähler parametrization are taken such that θ be real and z and $\bar{z}$ be complex conjugate to each other [8]. The complex functions parametrize a complex manifold whose metric has the Kähler potential $K(z, \bar{z})$ whose derivatives are denoted by $\partial K = \partial_z K$, $\overline{\partial} K = \partial_{\bar{z}} K$. The last term from the Lagrangian (1) is an arbitrary smooth function of $\rho = \sqrt{-j^2}$. The Euler-Lagrange equations derived from the Lagrangian (1) take the following form

$$\begin{aligned} & j_\mu \frac{df}{d\rho} - \rho(\partial_\mu \theta + i\partial K \partial_\mu z - i\overline{\partial} K \partial_\mu \bar{z}) = 0, \\ & \partial_\mu j^\mu = 0, \qquad 2i\partial\overline{\partial} K \, j_\mu \partial^\mu z = 0, \qquad 2i\partial\overline{\partial} K \, j_\mu \partial^\mu \bar{z} = 0. \end{aligned} \tag{2}$$

The action (1) is manifestly Lorentz invariant. Also, it is invariant under the space-time translations. It follows that the energy-momentum tensor is locally conserved

$$T_{\mu\nu} = g_{\mu\nu}\left(f'\sqrt{-j^2} - f\right) + f'\,\frac{j_\mu j_\nu}{\sqrt{-j^2}}, \qquad \partial^\mu T_{\mu\nu} = 0, \tag{3}$$

where f' is the derivative of $f(\rho)$ with respect to its argument. These equations are interpreted as fluid equations upon the identifications

$$\varepsilon = f(\rho), \qquad p = \rho f'(\rho) - f(\rho), \tag{4}$$

where p is the pressure and ε is the energy-density of the fluid.

Another important symmetry of the action is the invariance under the reparametrization of the complex surface of coordinates z and $\bar{z}$. There is an infinity number of reparametrization currents associated to this symmetry

$$J_\mu[G] = -2G(z,\bar{z})j_\mu, \qquad \partial^\mu J_\mu[G] = 0, \tag{5}$$

where $G(z,\bar{z})$ are arbitrary functions on the potentials. Finally, there is an axial-like symmetry of the action which has the conserved axial currents

$$k^\mu = \varepsilon^{\mu\nu\kappa\lambda}(\partial_\nu\theta + i\partial K\partial_\nu z - i\overline{\partial}K\partial_\nu\bar{z})\partial_\kappa(\partial_\lambda\theta + i\partial K\partial_\lambda z - i\overline{\partial}K\partial_\lambda\bar{z}), \qquad \partial_\mu k^\mu = 0, \tag{6}$$

which correspond to topological charges ω that are interpreted as being the linking numbers of vortices formed in the fluid

$$\omega = -2i\int d^3x\,\partial_i\left[\varepsilon^{ijk}\theta\partial\overline{\partial}K\partial_j\bar{z}\,\partial_k z\right]. \tag{7}$$

It is important to note that the components of the fluid current j^μ do not enter dynamically in the Lagrangian (1). Actually, j^μ is proportional do the velocity u^μ which represents the derivative with respect to the proper time along the fluid flow trajectories. This suggests that the components of the fluid current should not be considered true degrees of freedom. Indeed, since the fluid potentials have been used in the first place to parametrize the velocity, they are variables more fundamental than the currents [8]. They define a reduced set of variables from the initial configuration space.

In order to formulate the fluid dynamics in the Hamiltonian formalism, we calculate the canonical momenta associated to the Kähler parameters

$$\pi_\theta = \frac{\partial\mathcal{L}}{\partial\partial_0\theta} = j_0, \qquad \pi_z = \frac{\partial\mathcal{L}}{\partial\partial_0 z} = i\partial K j_0, \qquad \pi_{\bar{z}} = \frac{\partial\mathcal{L}}{\partial\partial_0\bar{z}} = -i\overline{\partial}K j_0. \tag{8}$$

According to the previous interpretation, the components of the currents do not enter the set of degrees of freedom. We conclude that the relevant phase space of the physical degrees of freedom is the *reduced phase space* [8] defined by the fields $\{\theta, z, \bar{z}, \pi_\theta, \pi_z, \pi_{\bar{z}}\}$. Indeed, treating j^μ either as degrees of freedom or Lagrange multipliers would lead us to the constraint that the velocity of the fluid (parametrized by the Kähler parameters) is zero. Thus, one should reformulate the above model in terms of the reduced phase space variables by expressing the components of the current in terms of the derivatives of θ, z and $\bar{z}$ from the equation of motion of j^μ.

One can see from the equations (8) that the fluid dynamics in the reduced phase space should obey the following second class constraints

$$\Omega_1 = \pi_z - i\partial K\pi_\theta = 0, \qquad \Omega_2 = \pi_{\bar{z}} + i\overline{\partial}K\pi_\theta = 0. \tag{9}$$

The Hamiltonian density calculated from the Lagrangian (1) has the form

$$H = \frac{\rho}{f'(\rho)}\delta_{mn}\left(\partial^m\theta + i\partial K\partial^m z - i\overline{\partial}K\partial^m\overline{z}\right)\left(\partial^n\theta + i\partial K\partial^n z - i\overline{\partial}K\partial^n\overline{z}\right) + f(\rho), \tag{10}$$

where $m, n = 1, 2, 3$ and $\rho^2 = \pi_\theta^2 - \delta_{mn}j^m j^n$

$$\rho^2 = \pi_\theta^2 - \frac{\rho}{f'(\rho)}\delta_{mn}\left(\partial^m\theta + i\partial K\partial^m z - i\overline{\partial}K\partial^m\overline{z}\right)\left(\partial^n\theta + i\partial K\partial^n z - i\overline{\partial}K\partial^n\overline{z}\right). \tag{11}$$

The Hamiltonian depends on the reduced phase space fields since the space-like components of the current are expressed in terms of these variables. The consequences of the constraints (9) to the classical theory were analyzed in [8]. In [14] the above results were derived by a different method and the system was quantized for the particular choice $K(z, \bar{z}) = z\bar{z}$ and $f(\rho) \sim \rho^2$ by applying the canonical quantization method.

3 BRST quantization in the reduced phase space

The relativistic fluid model from the previous section describes a large class of fluids parametrized by two arbitrary functions $K(z, \bar{z})$ and $f(\rho)$ which includes the perfect fluid, the fluids with power-law specific energies, Chaplygin gas, etc. Therefore, it is certainly interesting to find the quantum correspondents of these fluids. Since the classical dynamics of the relativistic fluid in the Kähler parametrization is governed by two second class constraints $\Omega_\alpha = \{\Omega_1, \Omega_2\}$, one can attempt to quantize this system by applying one of the methods designed to handle general situations of this type. In this section, we will use the Hamiltonian BRST formalism to calculate the invariant states and their time evolution. Since the states belong to a space that has an indefinite metric, we need to take into account the inner space structure to quantize the degrees of freedom. In this respect, our method is similar to the one proposed in [20, 21] and developed for gauge systems in [22, 23, 24, 25, 26, 27, 28, 29].

3.1 BRST invariant states

The perfect relativistic fluid in the Kähler parametrization can be quantized by applying the Hamiltonian BRST method [15]. As we have seen, the fluid potentials (fields) of the theory belong to the subspace Σ of the reduced phase space defined by the second class constraints $\Omega_\alpha = \{\Omega_1, \Omega_2\}$ given by the relations (9)

$$\Sigma : \{\theta, z, \bar{z}, \pi_\theta, \pi_z, \pi_{\bar{z}}, \Omega_\alpha\}.$$

Upon quantization, the potentials are elevated to operators acting on the Hilbert space of the quantum fluid and the constraints Ω_α become relations among operators. However, since the constraints are of second class their commutators do not vanish and one can easily show that they have the following form

$$[\Omega_1, \Omega_2] = 2\partial\bar{\partial}K j_0, \tag{12}$$

where we are using the supercommutator notation

$$[A, B] = AB - (-1)^{\varepsilon_A\varepsilon_B}BA, \tag{13}$$

with ε_A and ε_B the Grassmann parities of the operators A and B and $\varepsilon_A = 0, 1$ for A even and odd, respectively. Following [15], we extend the set of operators Σ by introducing a set of ghosts and anti-ghosts and their conjugate momenta in correspondence to each constraint

$$\Omega_\alpha \longrightarrow \left\{c^\alpha, p_\alpha, \bar{c}_\alpha, \bar{p}^\beta\right\}. \tag{14}$$

The fundamental commutators of the new operators are

$$[c^{\alpha}, p_{\beta}] = \delta^{\alpha}_{\beta}, \qquad [\bar{c}_{\alpha}, \bar{p}^{\beta}] = \delta^{\beta}_{\alpha}. \tag{15}$$

According to the general BRST method, the physical states are in the kernel of the nilpotent BRST operator

$$Q\,|\phi_{phys}\rangle = 0, \qquad Q^2 = 0. \tag{16}$$

However, from the relation (12) we can see that the BRST operator of the relativistic fluid which has the form of an abelian gauge theory

$$Q = c^{\alpha}\Omega_{\alpha}. \tag{17}$$

fails to be nilpotent

$$Q^2 = 2\partial\bar{\partial}K j_0 c^1 c^2. \tag{18}$$

In general, since $\partial\bar{\partial}K$ does not vanish on the constraint subspace, one should impose supplementary conditions in order to find the physical states. These conditions are: i) the conservation of the BRST charge and ii) the decomposition of it as

$$Q = \delta + \delta^{\dagger}, \qquad \delta^2 = 0, \qquad [\delta, \delta^{\dagger}] = 0. \tag{19}$$

If these relations are satisfied, then the physical states can be determined as the solutions of the following system [25, 22]

$$\delta\,|\phi_{phys}\rangle = \delta^{\dagger}\,|\phi_{phys}\rangle = 0, \tag{20}$$

The nontrivial solutions of the above equations can be written in the following form [24]

$$|\phi_{phys}\rangle = e^{[Q,\chi]}\,|\phi_{phys}\rangle_0\,, \tag{21}$$

where χ is a "gauge fixing" fermion with $gh(\chi) = -1$ and $|\phi_{phys}\rangle_0$ is a trivial BRST state determined by a complete irreducible set of BRST doublets in involution [27]. If the BRST operator is nilpotent and the theory has a Lie group gauge symmetry, the state $|\phi_{phys}\rangle$ is a BRST singlet. In the present case the BRST charge Q is not nilpotent according to (18). Therefore, the physical states are decomposed in higher multiplets than doublets. This decomposition can be performed by using the operator δ that can be determined by noting that the equations (19) are satisfied if two more sets of (dependent) ghost fields are associated to the constraints: g^{α} (complex and bosonic) and ψ_{α} (complex and fermionic). These new variables are specified by the following fundamental commutators among them

$$[g^{\alpha}, g^{\beta}] = 0, \qquad [\psi_1, \psi_2] = 0, \tag{22}$$

$$[g^{\alpha}, g^{\beta\dagger}] = 0, \qquad [\psi_{\alpha}, \psi^{\dagger}_{\beta}] = 2i\epsilon_{\alpha\beta}, \tag{23}$$

where $\epsilon_{\alpha\beta}$ is the anti-symmetric tensor with $\epsilon_{12} = 1$. From the above relations, it follows that the operator δ takes the following form

$$\delta = g^{\alpha}\psi_{\alpha}. \tag{24}$$

It is easy to see that the definition of the inner physical states given by the equations (20) can be recasted into the following set of equations

$$g^{\alpha}\,|\phi_{phys}\rangle = 0, \qquad \psi_{\alpha}\,|\phi_{phys}\rangle = 0. \tag{25}$$

These equations are not unique. Instead of them, one can use the following alternative equations which can be obtained by inverting the order of operators from the right hand side of (24)

$$g^{\alpha\dagger}\left|\phi_{phys}\right\rangle = 0, \qquad \psi_\alpha^\dagger \left|\phi_{phys}\right\rangle = 0. \tag{26}$$

As was proved in the general formulation of this method (see e.g. [28, 29]), there are different representations of the fields g^α and ψ_α in terms of constraints and ghosts. In particular, we can determine two of these representations that are useful for quantizing the relativistic fluid

$$g^\alpha = \frac{1}{2}(c^\alpha - i\bar{p}^\alpha), \qquad \psi_\alpha = \Omega_\alpha, \tag{27}$$

$$g^\alpha = \frac{1}{2}(c^\alpha + i\eta_\alpha\bar{p}^{\alpha+1}), \qquad \psi_\alpha = \Omega_\alpha, \tag{28}$$

where $\alpha + 1$ is taken $\bmod 2$ and $\eta_1 = 1, \eta_2 = -1$. By applying the general formalism from [27], we conclude that the operators that determine the physical states (21) can be organized in two doublets that contain the fields

$$D_1 : \bar{c}_2, \bar{p}^1, \qquad D_2 : \bar{c}_1, \bar{p}^2, \tag{29}$$

and two triplets of the following content

$$T_1 : c^2, p_1, \Omega_1, \qquad T_2 : c^1, p_2, \Omega_2. \tag{30}$$

Actually, the above sets of operators do not determine completely the BRST invariant states $\left|\phi_{phys}\right\rangle$ but only their phaseless component $\left|\phi_{phys}\right\rangle_0$. The requirement of the BRST involution imposes the constraint that the states $\left|\phi_{phys}\right\rangle_0$ be eigenstates corresponding to zero eigenvalues of all operators from involutions formed from pairs of doublets and triplets given by the relations (29) and (30). There are four such involutions denoted by $\{A\} = \overline{1,4}$ which consist of

$$\{1\} : \{D_1, T_1\}, \qquad \{2\} : \{D_2, T_2\}, \qquad \{3\} : \{D_2, T_1\}, \qquad \{4\} : \{D_1, T_2\}.$$

To each of $\{A\}$ corresponds a physical state $\left|\phi_{phys}, A\right\rangle_0$ defined by

$$\{A\}\left|\phi_{phys}, A\right\rangle_0 = 0, \tag{31}$$

For example, for $A = 1$ the state $\left|\phi_{phys}, 1\right\rangle_0$ is defined by the equations

$$\{1\}\left|\phi_{phys}, 1\right\rangle_0 \equiv \{D_1, T_1\}\left|\phi_{phys}, 1\right\rangle_0 = 0, \tag{32}$$

which is equivalent to the following five equations

$$\bar{c}_2\left|\phi_{phys}, 1\right\rangle_0 = \bar{p}^1\left|\phi_{phys}, 1\right\rangle_0 = 0, \tag{33}$$

$$c^2\left|\phi_{phys}, 1\right\rangle_0 = p_1\left|\phi_{phys}, 1\right\rangle_0 = \Omega_1\left|\phi_{phys}, 1\right\rangle_0 = 0. \tag{34}$$

The solutions $\left|\phi_{phys}, A\right\rangle$ can be obtained from (21) by specifying Λ_A in each $\{A\}$. One can show that the corresponding operators are

$$\Lambda_1 = i(c^1\bar{c}_1 - \bar{p}^2 p_2), \qquad \Lambda_2 = i(c^2\bar{c}_2 - \bar{p}^1 p_1), \tag{35}$$

$$\Lambda_3 = i(c^1\bar{c}_2 + \bar{p}^1 p_2), \qquad \Lambda_4 = -i(c^2\bar{c}_1 - \bar{p}^2 p_1). \tag{36}$$

The equations (21), (31), (35) and (36) determine completely the physical spectrum of the relativistic fluid in the Kähler parametrization. From them, we conclude that the states in each sector have the following form

$$|\phi_{phys}, A\rangle = e^{\Lambda_A} |\phi_{phys}, A\rangle_0 . \tag{37}$$

One can show that the states $|\phi_{phys}, A\rangle$ are BRST invariant singlets. Their ghost grading

$$gh(|\phi_{phys}, A\rangle) = \sum_{n=0} n(|\phi_{phys}, A\rangle) \tag{38}$$

guarantees that only $n = 0$ terms contributes to the inner products.

3.2 Unitary equivalent states

The physical states given by the relation (37) depend on the fermions χ_A since they are solutions of the equation (21). Due to this fact, fixing the form of the physical solutions is related to the morphisms of the operatorial space

$$\tilde{\Sigma} = \{\Omega_\alpha, c^\alpha, p_\alpha, \bar{c}_\alpha, \bar{p}^\alpha\}. \tag{39}$$

Note that by changing the operators χ_A or equivalently Λ_A, non-equivalent states $|\phi_{phys}, A\rangle$ can in principle be obtained [27]. However, in order to have a consistent theory, the norm of the physical states should be independent of the way in which the "gauge" χ_A is chosen.

The transformations of $\tilde{\Sigma}$ which are of interest are those that leave the BRST operator invariant since they guarantee that the physical states are defined by the invariant equation (16). In general, these transformations change the representation of δ - operator in terms of operators from $\tilde{\Sigma}$ and, consequently, transform the operators Λ_A. Since we have obtained our states in the representations given by the relations (27) and (28), respectively, we are interested in those morphisms of $\tilde{\Sigma}$ that leave the BRST operator invariant in these representations. The transformations that satisfy this property are endomorphisms that form the $U(1)^4$ group that have a two scale and a two rotation actions, respectively,

$$\begin{aligned} U_1(1) :& \Omega_\alpha \longrightarrow e^{\eta_\alpha \theta_1} \Omega_\alpha, \qquad c^\alpha \longrightarrow e^{-\eta_\alpha \theta_1} c^\alpha, \qquad p_\alpha \longrightarrow e^{\eta_\alpha \theta_1} p_\alpha, \\ U_2(1) :& \bar{c}_\alpha \longrightarrow e^{\eta_\alpha \theta_2} \bar{c}_\alpha, \qquad \bar{p}^\beta \longrightarrow e^{-\eta_\alpha \theta_2} \bar{p}^\beta, \end{aligned} \tag{40}$$

$$\begin{aligned} U_3(1) :& \begin{pmatrix} \Omega_1 \\ \Omega_2 \end{pmatrix} \longrightarrow \begin{pmatrix} \cos\theta_3 & \sin\theta_3 \\ -\sin\theta_3 & \cos\theta_3 \end{pmatrix} \begin{pmatrix} \Omega_1 \\ \Omega_2 \end{pmatrix}, \qquad \begin{pmatrix} c^1 \\ c^2 \end{pmatrix} \longrightarrow \begin{pmatrix} \cos\theta_3 & \sin\theta_3 \\ -\sin\theta_3 & \cos\theta_3 \end{pmatrix} \begin{pmatrix} c^1 \\ c^2 \end{pmatrix}, \\ & \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} \longrightarrow \begin{pmatrix} \cos\theta_3 & \sin\theta_3 \\ -\sin\theta_3 & \cos\theta_3 \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \end{pmatrix}, \\ U_4(1) :& \begin{pmatrix} \bar{p}^1 \\ \bar{p}^2 \end{pmatrix} \longrightarrow \begin{pmatrix} \cos\theta_4 & \sin\theta_4 \\ -\sin\theta_4 & \cos\theta_4 \end{pmatrix} \begin{pmatrix} \bar{p}^1 \\ \bar{p}^2 \end{pmatrix}, \qquad \begin{pmatrix} \bar{c}_1 \\ \bar{c}_2 \end{pmatrix} \longrightarrow \begin{pmatrix} \cos\theta_4 & \sin\theta_4 \\ -\sin\theta_4 & \cos\theta_4 \end{pmatrix} \begin{pmatrix} \bar{c}_1 \\ \bar{c}_2 \end{pmatrix}, \end{aligned} \tag{41}$$

where $\theta_i, i = \overline{1,4}$ are real parameters. The representations relations (27) and (28) are not unique, nor they are exhausted by the transformations from $U(1)^4$. Nevertheless, once a different representation from (27) and (28) is chosen, the representations that are equivalent with it in the sense that the BRST operator is invariant, are connected by $U(1)^4$ transformations.

The action of the transformations (27) and (28) on the operators Λ_A is given by the following relations

$$\Lambda_A \longrightarrow e^{\Theta_A} \Lambda_A, \tag{42}$$

where

$$\Theta_1 = \theta_2 - \theta_1, \Theta_2 = \theta_1 - \theta_2, \Theta_3 = -(\theta_1 + \theta_2), \Theta_4 = \theta_1 + \theta_2. \tag{43}$$

Due to the presence of the phases Θ_A the physical states obtained by scale transformations are not unitarily equivalent. However, this shortcoming can be circumvented by dividing the physical states by these phases

$$|\phi_{phys}, A\rangle = e^{-\frac{\Theta_A}{2} + \Lambda_A} |\phi_{phys}, A\rangle_0 . \tag{44}$$

The states given by the relation (44) represent the physical states of the relativistic fluid in the Kähler parametrization that are unitarily invariant under the scaling of the operatorial set $\tilde{\Sigma}$. These solutions are similar to the ones of the abelian field theory. As a matter of fact, the constraints and the reduced phase space have an abelian structure from the beginning, once the components of the current are expressed in terms of fluid potentials. Along the line of this interpretation, the states $|\phi_{phys}, A\rangle$ describe quantum fluctuations of the fluid potentials around the current flows. More general solutions can be obtained from $|\phi_{phys}, A\rangle$ by noting that the Λ_A operators form a closed algebra [27]

$$|\phi_{phys}, A\rangle' = [2\left(\frac{b}{a}\right)^{\frac{1}{2}} \sinh(2(|ab|^{\frac{1}{2}})]^{\frac{1}{2}} e^{a\Lambda_1 + b\Lambda_2} |\phi_{phys}, A\rangle_0 , \qquad A = 1, 2, \tag{45}$$

$$|\phi_{phys}, A\rangle' = [2\left(\frac{b}{a}\right)^{\frac{1}{2}} \sinh(2(|ab|^{\frac{1}{2}})]^{\frac{1}{2}} e^{a\Lambda_3 + b\Lambda_4} |\phi_{phys}, A\rangle_0 , \qquad A = 3, 4, \tag{46}$$

where a and b are real parameters. In the case of gauge theories, these states belong to the subspace $\mathcal{H}_{phys}$ of the nondegenerate inner vector space associated to the BRST operator. Since our states are singlets, $\mathcal{H}_{phys}$ can in principle be determined by performing the Hodge decomposition with respect to the coBRST charge [31] or by decomposing each multiplet in terms of ghosts [27]. These methods can be applied to the quantum fluid, too.

4 Dynamics of the quantum relativistic fluid

The states determined in the previous section can be generalized to non-stationary states. To this end, we will find the BRST invariant Hamiltonian. Next, we are going to use the states from states (45) and (46) as a starting point to determine the time evolution and the path integral of the quantum fluid.

4.1 Time evolution of physical states

According to [29, 30], the dynamics of an arbitrary gauge system is defined by the time evolution of the physical states

$$i\frac{\partial}{\partial t} |\phi_{phys}(t)\rangle = H_{BRST} |\phi_{phys}(t)\rangle , \tag{47}$$

where the operator H_{BRST} is BRST invariant. In general, the operator H_{BRST} of an arbitrary gauge theory is not unique, but rather belongs to a set of operators. These BRST invariant Hamiltonians can be obtained from the original Hamiltonian by adding terms that are polynomial in ghosts. However, if the following conditions are satisfied: i) the physical states are singlets under the BRST operator, ii) the operator H_{BRST} satisfies the following relation

$$[H_{BRST}, [Q, \chi]] = 0, \tag{48}$$

and iii) the gauge fixing functions are linear in time, then a certain algorithm for picking up the correct H_{BRST} from the BRST invariant Hamiltonians can be given. Moreover, by time slicing the time evolution of the physical states, one can construct the path integral of probability amplitudes of the system [29, 30].

In the case of the quantum relativistic fluid discussed in the previous section, the physical states are BRST singlets. It follows that the time dependent BRST invariant states should have the following form

$$|\phi_{phys}, A(t)\rangle = e^{-itH_{BRST}} |\phi_{phys}, A\rangle . \tag{49}$$

The consistency conditions can be summarized in this case by the following set of equations

$$[Q, H_{BRST}] = [\Lambda_A, H_{BRST}] = 0. \tag{50}$$

The first of the above equations imposes the BRST invariance of the Hamiltonian while the second one guarantees that the states $|\phi_{phys}, A(t)\rangle_0$ evolve with the same time evolution operator H_{BRST}. By using the relations (10), (17), (35) and (36) into the equations (50), one can see that the original Hamiltonian H satisfy the above conditions. Therefore, the natural choice for the BRST invariant operator is

$$H_{BRST} = H. \tag{51}$$

The dynamics of the quantum states of the relativistic fluid is specified by the relations (49) and (50). These equations can be used to derive the path integral formulation of quantum fluid.

4.2 Path integral formulation

As the previous analysis has revealed, the physical inner states of (21) belong to the inner space. (Like in the case of the gauge systems, one would not expect that $|\phi_{phys}, A\rangle_0$ be from the inner space but only its zero ghost term). Since the time evolutions of these states is given by the equation (49) where the Hamiltonian satisfies (50) one can use their probability amplitude to define the path integral similarly to the case of the abelian gauge theory [30]. To this end, we take the states from (45) and compute their amplitude as they evolve in time between t_1 and t_2

$${}'\langle\phi_{phys}, A(t_1)|\phi_{phys}, B(t_2)\rangle' = {}_0\left\langle\phi_{phys}, A|e^{i(t_1-t_2)H+2a\Lambda_1+2b\Lambda_2}|\phi_{phys}, B\right\rangle_0 \delta_{AB}. \tag{52}$$

The above relation can be written as a path integral in terms of an effective Hamiltonian if one rescale the real coefficients as

$$a \longrightarrow \frac{1}{2}(t_1 - t_2)a, \qquad b \longrightarrow \frac{1}{2}(t_1 - t_2)b. \tag{53}$$

Then one can show that the path integral takes the following form

$$\langle\Phi(t_1)|\Phi(t_2)\rangle = \int D[\Phi]D[\Pi] \exp\left\{i\int_{t_1}^{t_2} dt' \left[\Pi_a(t')\frac{d\Phi^a(t')}{dt'} - H_{eff}(\Phi^a(t'), \Pi_a(t'))\right]\right\}. \tag{54}$$

Here, we have introduced the notation

$$\Phi : \{\theta, z, \bar{z}, c^\alpha, \bar{c}_\alpha\}, \qquad \Pi : \{\pi_\theta, \pi_z, \pi_{\bar{z}}, p_\alpha, \bar{p}^\alpha\}. \tag{55}$$

The effective Hamiltonian is given by the following relation

$$H_{eff} = \frac{\rho}{f'(\rho)} \left(\partial^m \theta + i\partial K \partial^m z - i\overline{\partial} K \partial^m \overline{z}\right) \left(\partial_m \theta + i\partial K \partial_m z - i\overline{\partial} K \partial_m \overline{z}\right) + f(\rho) + a\bar{\Lambda}_1 + b\bar{\Lambda}_2, \tag{56}$$

where $\bar{\Lambda}_1$ and $\bar{\Lambda}_2$ are classical counterparts of the corresponding operators. The effective Hamiltonian depends on the choice of these operators as well as on the factorization (27) and (28) of the δ - operator. In the case of gauge theories, the effective Hamiltonian obtained in this way does not lead in general to a regular Lagrangian. This can be obtained by considering the full algebraic structure of the operators Λ_1 and Λ_2. Finally, note that the path integral in the $A = 3, 4$ sector can be obtained in the same way.

5 Discussions

In this paper we have quantized the relativistic fluid by using the Hamiltonian BRST method in the reduced phase space of the fluid potentials and their canonical conjugate momenta. This space is subjected to second class constraints which make it similar to the phase space of some gauge field theories. However, some differences should be noted. By solving the components of the fluid current in terms of the Kähler parameters as in [8], the system lacks first class constraints that are present in field theories. As a consequence, there is no longer a full gauge structure present in the configuration space. This has two formal consequences. The first one is that there are fields missing from the BFV scheme, namely the ones associated to the Lagrange multipliers which, in the end, produce delta functions for the constraints in any possible prescription that is obtained by rotating an effective Hamiltonian and the corresponding δ - operator by transformations from $U(1) \times U(1)$ subgroup of $U(1)^4$. Secondly, the BRST invariant Hamiltonian is uniquely determined by the original Hamiltonian of the fluid.

By exploiting the similarities between the reduced phase space of the fluids and the phase space of the gauge theories, we have obtained the physical states as singlets of the BRST operator in (44). These states have been used to derive the effective Hamiltonian given by the relation (56). However, this type of Hamiltonian does not lead to regular Lagrangians in gauge theories and it is not gauge fixed. In the case of the quantum fluid, we can remedy this situation by choosing the representation

$$g^1 = \frac{1}{2}(c^1 + ic^2), \qquad \psi_1 = \Omega_1 + i\Omega_2, \tag{57}$$

$$g^2 = \frac{1}{2}(p^1 + ip^2), \qquad \psi_2 = 0, \tag{58}$$

which lead to the following effective Hamiltonian

$$H'_{eff} = \frac{\rho}{f'(\rho)} \left(\partial^m \theta + i\partial K \partial^m z - i\overline{\partial} K \partial^m \overline{z}\right) \left(\partial_m \theta + i\partial K \partial_m z - i\overline{\partial} K \partial_m \overline{z}\right) + ic^1 p_2 + i\bar{p}^1 \bar{c}_2. \tag{59}$$

The absence of the $\bar{\Lambda}_1$ and $\bar{\Lambda}_2$ functions indicates the analogue of the gauge fixing from the gauge field.

From the results obtained in this paper we see that it is important to address the quantization of the relativistic fluids in terms of the full set of variables. This could provide a more deep understanding of their formal and physical properties.

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